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On some infinite dimensional linear groups

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Abstract. Let F be a field and A an (infinite dimensional) vector space over F. A group G of linear transformations of A is said to be *finitary linear* if for each element $g \in G$ the centralizer $C_A(g)$ has finite codimension over F. Finitary linear groups are natural analogs of FC-groups (i.e. groups with finite conjugacy classes). In this paper we consider linear analogs of groups with boundedly finite conjugacy classes, and also some generalizations corresponding to groups with Chernikov conjugacy classes.

Keywords: Finitary linear group, FC-group, artinian-finitary module.

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Let F be a field and A a vector space over F. Denote by $GL(F, A)$ the group of all F-automorphisms of A. The subgroups of $GL(F, A)$ are called the *linear groups*. Linear groups play a very important role in algebra and other branches of mathematics. If $dim_F(A)$ (the dimension of A over F) is finite, say n, then a subgroup G of $GL(F, A)$ is a *finite dimensional* linear group. It is well known that in this case, $GL(F, A)$ can be identified with the group of all invertible $n \times n$ matrices with entries in F. The theory of finite dimensional linear groups is one of the most developed in group theory. It uses not only algebraic, but also topological, geometrical, combinatorial, and many other methods.

However, in the case when A has infinite dimension over F, the study of the subgroups of $GL(F, A)$ requires some additional restrictions. This case is more complicated and requires some additional restrictions allowing an effective employing of already developed techniques. The most natural restrictions here are the finiteness conditions. Finitary linear groups demonstrate the efficiency of such approach. We recall that a subgroup G of $GL(F, A)$ is called *finitary* if for each element $g \in G$ its centralizer $C_A(g)$ has finite codimension over F. The theory of finitary linear groups is now well-developed and many interesting results have been proved (see, for instance, the survey [1]). We begin with consideration on some generalizations of such groups.

1 On some generalizations of finitary linear groups

If G is a subgroup of $GL(F, A)$, we can consider the vector space A as a module over the group ring FG. We can obtain the following generalizations of finitary groups. Replacing the field F by the ring R, artinian and noetherian R-modules are natural generalizations of the concept of a finite dimensional vector space. Some related generalizations of finitary groups have been considered by B.A.F. Wehrfritz (see [2], [3], [4], [5]).

Let R be a ring, G a group and A an RG-module. Following B.A.F. Wehrfritz, a group G is called *artinian - finitary*, if for every element $g \in G$, the factor-module $A/C_A(g)$ is artinian as an R-module. In this case, we say that A is an *artinian - finitary* RG-module.

We observe that we can consider finitary linear groups as linear analogs of the FC-groups (we can define an FC-group G as a group such that $|G : C_G(x)|$ is finite for each element $g \in G$). Similarly, if $R = \mathbb{Z}$ and G is an artinian - finitary group, then the additive group of the factor-module $A/C_A(g)$ is Chernikov for every element $g \in G$. This shows that we can consider artinian - finitary groups as linear analogs of the groups with Chernikov conjugacy classes (shortly CC-groups).

One of the first important result of theory of FC-groups was a theorem due to B. H. Neumann that described the structure of FC- groups with bounded conjugacy classes. Following B. H. Neumann, a group G is called a BFC-group if there exists a positive integer b such that $|g^G| \leq b$ for each element $g \in G$. B. H. Neumann proved that a group G is a BFC-group if and only if the derived subgroup $[G, G]$ is finite ([6], Theorem 3.1).

A group $G \leq GL(F, A)$ is said to be a *bounded finitary linear group*, if there is a positive integer b such that $dim_F A/C_A(g) \leq b$ for each element $g \in G$. These groups are some linear analogs of BFC-groups. Let ωRG be the *augmentation ideal* of the group ring RG, i.e. the two-sided ideal of RG generated by the all elements $g - 1,\ g \in G$. The submodule $A(\omega FG)$ is called the derived submodule. We can consider the derived submodule as a linear analog of the derived subgroup. Note that in the general case we cannot obtain an analog of Neumann's theorem. It is not hard to construct an F_pG-module A over an infinite elementary abelian group G such that G is bounded finitary linear group but $A(\omega F_pG)$ has infinite dimension over F_p (see [7]). However, under some natural restrictions on the p-sections of a bounded finitary linear group, the finiteness of $dim_F(A(\omega FG))$ can be proved. Thus some linear analog of B. H. Neumann's theorem can be established. We considered a more general situation.

Let A be an artinian $\mathbb{Z}$-module. Then a set $\Pi(A)$ is finite. If D is a divisible part of A, then $D = K_1 \oplus \ldots \oplus K_d$ where K_j is a Prüfer subgroup, $1 \leq J \leq d$. The number d is an invariant of A. Another important invariant here is the order of A/D.

If D is a Dedekind domain, the structure of the artinian D-module A is very similar to that described above. Let D be a Dedekind domain. Put

$$Spec(D) = \{P\,|P \text{ is a maximal ideal of } D\}$$

Let P be a maximal ideal of D. Denote by A_P the set of all elements a such that $Ann_D(a) = P^n$ for some positive integer n. If A is a D-periodic module, then define

$$Ass_D(A) = \{P \in Spec(D)|A_P \neq < 0 >\}.$$

In this case, $A = \oplus_{P\in\pi} A_P$ where $\pi = Ass_D(A)$ (see, for instance, [8], Corollary 6.25). If A is an artinian D-module, then A is D-periodic and the set $Ass_D(A)$ is finite. Furthermore, $A = K_1 \oplus \cdots \oplus K_d \oplus B$ where K_j is a Prufer submodule, $1 \leq j \leq d$, B is a finitely generated submodule (see, for instance, [9], Theorem 5.7). Here the Prufer submodule is a D-injective evelope of a simple submodule. Observe that this decomposition is unique up to isomorphism. It follows that the number d is an invariant of the module A. Put $d = I_D(A)$. The submodule B has a finite series of submodules with D - simple factors. The Jordan-Holder Theorem implies that the length of this composition series is also an invariant of B, and hence of A. Denote this number by $I_F(A)$.

Let D be a Dedekind domain and G a group. The DG-module A is said to be a *bounded artinian finitary* if A is artinian finitary and there are positive integers b and d and a finite subset $\tau \subseteq Spec(D)$ such that $I_F(A/C_A(g)) \leq b$, $I_D(A/C_A(g)) \leq d$ and $Ass_D(A/C_A(g)) \subseteq b_\sigma(A)$. We will use the following notation:

$$\pi(A) = \{p|p = charD/P \quad \text{for all} \quad P \in b_\sigma(A)\}.$$

The group G is said to be *generalized radical* if G has an ascending series whose factors are either locally nilpotent or locally finite. Let p be a prime. We say that a group G has *finite section p-rank* $r_p(G) = r$ if every elementary abelian p-section U/V of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

In the paper [10], the following analog of Neumann's theorem has been obtained.

Theorem 1. (L.A.Kurdachenko, I.Ya.Subbotin, V.A.Chepurdya [10]) *Let D be a Dedekin domain, G a locally generalized radical group, and A a DG-module. Suppose that A is a bounded artinian finitary module. Assume also that there exists a positive integer r such that the section p-rank of G is at most r for all $p \in \pi(A)$. Then* *a) the submodule $A(\omega FG)$ is artinian as a D-module,*

b) the factor-group $G/C_G(A)$ has finite special rank.

Corollary 1. *Let F be a field, A a vector space over F, G a locally generalized radical subgroup of $GL(F, A)$. Suppose that there exists a positive integer r such that the section p-rank of G is at most r were $p = charF$. Then*

a) the submodule $A(\omega FG)$ is finite dimensional,

b) the factor-group $G/C_G(A)$ has finite special rank.

As we noted above the restriction on the section p-rank is essential.

2 Linear groups that are dual to finitary

Consider another analog of FC-groups which is dual in some sense to finitary linear groups. We introduce this concept not only for linear groups, but in a more general situation.

Let R be a ring, G a group and A an RG-module. If a is an element of A, then the set

$$aG = \{ag|g \in G\}$$

is called *the G-orbit of a.*

We say that G *has finite orbits on A* if the orbit aG is finite for all $a \in A$.

By the orbit stabilizer theorem, it is clear that in this situation, $|aG| = |G : C_G(a)|$ is finite, so we can think of aG as the analog of a conjugacy class.

Let F be a field and let G be a subgroup of $GL(F, A)$. Suppose that $dim_F(A)$ is finite and choose a basis $a_1, ..., a_n$ for the vector space A. Suppose that G has finite orbits on A. Then every element of $C_G(a_1) \cap ... \cap C_G(a_n)$ acts trivially on A, and hence $C_G(a_1) \cap ... \cap C_G(a_n) =< 1 >$. However, this intersection has finite index in G and hence G is finite. Thus, we can think of linear groups with finite orbits as generalizations of finite groups.

We say that G has *boundedly finite orbits on A* if there is a positive integer b such that $|aG| \leq b$ for each element $a \in A$. The smallest such b will be denoted by $lo_A(G)$.

Since $|aG| = |G : C_G(a)|$ for all $a \in A$, it is not hard to see that any group G in which $G/C_G(A)$ is finite has boundedly finite orbits on A. However, as the following example shows, the converse statement is far from being true.

Let A be a vector space over the field F admitting the basis $\{a_n|n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ the mapping $g_n : A \longrightarrow A$, given by

$$a_m g_n = \begin{cases} a_1 + a_m & \text{if } m = n + 1 \\ a_m & \text{if } m \neq n + 1 \end{cases}$$

is an F-automorphism of A. Then $G = \langle g_n | n \in \mathbb{N} \rangle$ is a subgroup of $GL(F, A)$. Clearly $[g_n, g_m] = 1$ whenever $n \neq m$, so that G is abelian. Moreover, if $char F = p > 0$, then G is an elementary abelian p-group. It follows in this case that $ag = a + ta_1$ for every $a \in A$, where $0 \leq t < p$. Consequently,

$$aG = \{a, a + a_1, a + 2a_1, ..., a + (p - 1)a_1\}.$$

Therefore, $|aG| \leq p$ for each element $a \in A$, and G has boundedly finite orbits on A. However, it is clear that $C_G(A) =< 1 >$, so that $G/C_G(A)$ is infinite.

Let B be a vector space over a field F of characteristic $p > 0$ admitting the basis $\{b_n | n \in \mathbb{N}\}$. We define the mapping $x : B \longrightarrow B$ by the rule

$$b_m x = \begin{cases} b_m & \text{if } m \text{ is even} \\ b_{2n} + b_{2n+1} & \text{if } m = 2n + 1. \end{cases}$$

Clearly, x is an F-automorphism of B and $B(\omega F < x >) = \oplus_{n \in \mathbb{N}} b_{2n} F$. In particular, the dimension of $B(\omega F < x >)$ is infinite. Since $|x| = p$, $|b < x >| \leq p$ for each element $b \in B$. Now let A and G be the vector space and the linear group from the first example, respectively. Then $L = G\times < x >$ acts on the vector space $C = A \oplus B$ in the natural way. Clearly, $|cL| \leq p^2$ for every element $c \in C$. However, the factor-group $L/C_L(C)$ is infinite and the dimension of $C(\omega FL)$ is infinite. In other words, we cannot have an analog of Neumann's theorem.

Next result describes linear groups acting with boundedly finite orbits.

Theorem 2. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, R a ring and A an RG-module. Suppose that G acts on A with boundedly finite G-orbits, and let $b = lo_A(G)$. Then*

i) $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite exponent such that G/L is finite.

ii) A contains an RG-submodule C such that C is finitely generated as an R-module and L acts trivially on C and A/C.

iii) There is a positive integer m such that m is a divisor of $b!$ and $mA(\omega RG) =< 0 >$.

Note that in the above statement the submodules of C need not be finitely generated. Therefore, we cannot deduce in this theorem that $A(\omega RG)$ is finitely generated as an R-module. However, if R is noetherian, then every finitely generated R-submodule is also noetherian. So in this case, every submodule of C is finitely generated. Even when R is a noetherian ring so that $A(\omega RG)$ is finitely generated, in general it appears that nothing can be deduced concerning its number of generators. We can now establish our next main theorem.

Theorem 3. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, R a noetherian ring and A an RG-module.*

i) Suppose that G acts on A with boundedly finite G-orbits, and let $b = lo_A(G)$. Then $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finitely generated.

ii) If a factor-group $G/C_G(A)$ has a normal subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finite, then G has boundedly finite orbits on A.

iii) If there is an integer b such that $R/b!\ R$ is finite and $b = lo_A(G)$, then $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite index and finite exponent such that $A(\omega RG)$ is finite.

Next we give some specific examples of rings satisfiyng the conditions of Theorem 3. Of course, one particular interesting example is the ring of integer.

Corollary 2. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group acting on the $\mathbb{Z}G$-module A. Then G has boundedly finite orbits on A if and only if G contains a normal subgroup L such that G/L and $A(\omega\mathbb{Z}L)$ are finite.*

Next result is a generalization of Corollary 2. An infinite Dedekin domain D is said to be a *Dedekind $\mathbb{Z}_0$-domain* if for every maximal ideal P of D, the quotient ring D/P is finite (see for instance [9], Chapter 6). If F is a finite field extension of $\mathbb{Q}$ and R is a finitely generated subring of F, then R is an example of a Dedekind $\mathbb{Z}_0$ domain.

Corollary 3. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, D a Dedekind $\mathbb{Z}_0$-domain and A a DG-module. Then G has boundedly finite orbits on A if and only if there exists a normal abelian subgroup $L/C_G(A)$ of $G/C_G(A)$ of finite index and finite exponent such that $A(\omega DG)$ is finite.*

For the case when the ring of scalars is a field, we obtain

Theorem 4. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, F a field of characteristic $p > 0$ and A an FG-module. Suppose that G acts on A with boundedly finite G-orbits. Then*

i) $G/C_G(A)$ contains a normal abelian p-subgroup $L/C_G(A)$ of finite exponent such that G/L is finite.

ii) A contains an FG-submodule C such that $dim_F(C)$ is finite and L acts trivially on C and A/C.

Next result deals with the situation when $G/C_G(A)$ is finite.

Theorem 5. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, F a field and A an FG-module. Suppose that G acts on A with boundedly finite G-orbits. Assume that if $charF = p > 0$, then $G/C_G(A)$ is a p'-group. Then $G/C_G(A)$ is finite.*

In particular, if F is a field of characteristic 0, then G acts on the FG-module A with boundedly finite G-orbits if and only if $G/C_G(A)$ is finite.

We consider now the following generalization. If a group G acts on A with finite G-orbits, then an FG-submodule aFG has finite dimension over F.

Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. We say that G is a *linear group with finite dimensional G-orbits* (or that *A has finite dimensional G-orbits*) if the G-orbit aG generates a finite dimensional subspace for each element $a \in A$.

As we have seen above, if a group G has finite G-orbits then G has finite dimensional G-orbits, but the converse is false. Every ordinary finite dimensional linear group G is a group with finite dimensional G-orbits. But we have seen above that if a finite dimensional linear group G has finite G-orbits, then G is finite.

We say that a linear group G *has boundedly finite dimensional orbits on A* if there is a positive integer b such that $dim_F(aFG) \leq b$ for each element $a \in A$. Put

$$md(G) = max\{dim_F(aFG) \quad | \quad a \in A\}.$$

Every linear group G defined over a finite dimensional vector space A is a group with boundedly finite dimensional orbits.

In view of Neumann's result, a natural question arises: when is $dim_F(A(\omega FG))$ finite? An easy computation shows that $aFG \leq A(\omega FG) + aF$ for each $a \in A$, and hence if $dim_F(A(\omega FG)) \leq d$ then aFG is of F-dimension at most $d + 1$. Thus, if $A(\omega FG)$ is finite dimensional, then G has boundedly finite dimensional orbits. However, as we showed above, even for linear groups having boundedly finite orbits on A, the converse is false. It would be interesting to know which conditions imposed on a group G implies that $A(\omega FG)$ is finite dimensional.

Let B be a subspace of A, then the *norm of B* in G is the subgroup

$$Norm_G(B) = \bigcap_{b \in B} N_G(bF).$$

Observe that $Norm_G(B)$ is the intersection of the normalizers of all F-subspaces of B, and that $G = Norm_G(A)$ if and only if every subspace of A is G-invariant.

The following theorem provides us with a description of linear groups having boundedly finite dimensional orbits on A.

Theorem 6. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Suppose that G has boundedly finite dimensional orbits on A and let $b = md(G)$. Then*

i) A has an FG-submodule D such that $dim_F(D)$ is finite and if $K = C_G(D)$, then $K \leq Norm_G(A/D)$. Moreover there exists a function f such that $dim_F(D) \leq f(b)$.

ii) K is a normal subgroup of G and has a G-invariant abelian subgroup T such that $A(\omega FT) \leq D$ and K/T is isomorphic to a subgroup of the multiplicative group of a field F.

iii) T is an elementary abelian p-subgroup if $charF = p > 0$ and is a torsion-free abelian group otherwise.

In particular, G is an extension of a metabelian group by a finite dimensional linear group.

We use Theorem 6 to establish several properties of groups with boundedly finite dimensional orbits that are analogs to corresponding results for finite dimensional linear groups. There are many applications of Theorem 6. Here we just select some of them. It is a well-known theorem of Schur that periodiic finite dimensional linear groups are locally finite.

Corollary 4. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Suppose that G has boundedly finite dimensional orbits on A.*

i) If G is periodic then G is locally finite.

ii) If G is locally generalized radical then G is locally (finite and soluble).

iii) If G is a periodic p'-group, where $p = charF$, then the centre of G includes a locally cyclic subgroup K such that G/K is soluble-by-finite.

Now we consider another topic: the reduction to the groups with finite dimensional orbits.

Let again G be a subgroup of $GL(F, A)$. We say that G is a *linear group with finite G-orbits of subspaces* if the set $cl_G(B) = \{Bg \quad | \quad g \in G\}$ is finite for each F-subspace B of A. Groups with this property are natural analogs of groups with finite G-orbits of elements. Since it is clear that $|cl_G(B)| = |G : N_G(B)|$, it follows that G has finite G-orbits of subspaces if and only if the indexes $|G : N_G(B)|$ are finite for all F-subspaces B of A. It is not hard to prove that if G has finite G-orbits of subspaces then $dim_F(aFG)$ is finite, for each element $a \in A$.

Observe that if every F-subspace B is G-invariant, then G is abelian. Linear groups with finite G-orbits of subspaces can be considered as natural generalizations of abelian linear groups.

For these groups we obtain the following result.

Theorem 7. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Suppose that G is a linear group with finite G-orbits of subspaces. Then the factor group $G/Norm_G(A)$ is finite and G is central-by-finite.*

We say that a group has *boundedly finite G-orbits of subspaces* if there is a positive integer b such that $|cl_G(B)| \leq b$ for all subspaces B of A.

Corollary 5. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Then G has finite G-orbits of subspaces if and only if G has boundedly finite G-orbits of subspaces.*

3 Linear groups with restriction on subgroups of infinite central dimension

If H is a subgroup of $GL(F, A)$, then H really acts on the factor-space $A/C_A(H)$. Following [13] we say that H has *finite central dimension*, if $dim_F(A/C_A(H))$ is finite. In this case $dim_F(A/C_A(H)) = centdim_F(H)$ will be called the *central dimension of the subgroup H*.

If H has finite central dimension, then $A/C_A(H)$ is finite dimensional. Put $C = C_G(A/C_A(H))$. Then, clearly, C is a normal subgroup of H and H/C is isomorphic to some subgroup of $GL_n(F)$ where $n = dim_F(A/C_A(H))$. Each element of C acts trivially on every factor of the series $< 0 >\leq C_A(H) \leq A$, so that C is an abelian subgroup. Moreover, if $charF = 0$, then C is torsion-free; if $charF = p > 0$, then C is an elementary abelian p-subgroup. Hence, the structure of H in general is defined by the structure of G/C, which is an ordinary finite dimensional linear group.

Let $G \leq GL(F, A)$ and let $L_{icd}(G)$ be the set of all proper subgroups of G having infinite central dimension. In the paper [13], it has been proved that if every proper subgroup of G has finite central dimension, then either G has finite central dimension or G is a Prufer p-group for some prime p (under some natural restrictions on G). This shows that it is natural to consider those linear groups G, in which the family $L_{icd}(G)$ is "very small" in some particular sense. But what means "very small" for infinite groups? One of the natural approaches possible here is to consider finiteness conditions. More precisely, it is natural to consider the groups in which the family $L_{icd}(G)$ satisfies a suitable strong finiteness condition. In the paper [14] we considered some of such situations. In particular, linear groups in which the family $L_{icd}(G)$ satisfies either the minimal or the maximal condition and some rank restriction were considered. The weak minimal and weak maximal conditions are natural group-theoretical generalizations of the ordinary minimal and maximal conditions. These conditions have been introduced by R.Baer [15] and D.I.Zaitsev [16]. The definition of the weak minimal condition in the most general form is the following.

Let G be a group and $\mathcal{M}$ a family of subgroups of G. We say that $\mathcal{M}$ satisfies the *weak maximal* (respectively *minimal*) condition (or that G satisfies the *weak maximal* (respectively *minimal*) condition for $\mathcal{M}$-subgroups), if for every ascending (respectively descending) chain $\{H_n \mid n \in \mathbb{N}\}$ of subgroups in the family M there exists a number $m \in \mathbb{N}$ such that the indexes $|H_{n+1} : H_n|$ (respectively $|H_n : H_{n+1}|$) are finite for all $n \geq m$.

Groups with the weak minimal or maximal conditions for some important families of subgroups have been studied by many authors (see, for instance, the book [17],5.1, and the survey [18]).

We say that a *group $G \leq GL(F, A)$ satisfies the weak maximal (respectively minimal) condition for subgroups of infinite central dimension, or shortly $Wmax - icd$ (respectively $Wmin - icd$)*, if the family $L_{icd}(G)$ satisfies the weak maximal (respectively minimal) condition.

The first results about linear groups satisfying the conditions $Wmin - icd$ and $Wmax - icd$ have been obtained in [19]. More precisely, this paper was devoted to the study of periodic groups with such properties. The main results are the following

Theorem 8. (J.M. Munoz-Escolano, J. Otal, N.N. Semko [19]) *Let F be a field, A a vector space over F and G a locally soluble periodic subgroup of $GL(F, A)$. Suppose that G has infinite central dimension and satisfies $Wmin - icd$ or $Wmax - icd$. The following assertions hold*

1) *If $char F = 0$, then G is a Chernikov group.*

2) *If $char F = p > 0$, then either G is a Chernikov group or G has a series of normal subgroups $H \leq D \leq G$ satisfying the following conditions:*

   2a) *H is a nilpotent bounded p-subgroup.*

   2b) *$D = H\lambda Q$ for some non-identity divisible Chernikov subgroup Q such that $p \notin \Pi(Q)$.*

   2c) *H has finite central dimension, Q has infinite central dimension.*

   2d) *If K is a Prufer q-subgroup of Q and K has infinite central dimension, then H has a finite K-composition series.*

   2e) *G/D is finite.*

Corollary 6. (J.M. Munoz-Escolano, J. Otal, N.N. Semko [19]) *Let F be a field, A a vector space over F and G a locally soluble periodic subgroup of $GL(F, A)$. Then the following conditions are equivalent:*

i) G satisfies the weak minimal condition on subgroups of infinite central dimension;

ii) G satisfies the weak maximal condition on subgroups of infinite central dimension;

iii) G satisfies the minimal condition on subgroups of infinite central dimension.

Corollary 7. (J.M.Munoz-Escolano, J.Otal, N.N.Semko [19]) *Let F be a field, A a vector space over F and G a locally nilpotent subgroup of $GL(F, A)$. Suppose that G has infinite central dimension. Then the following conditions are equivalent:*

i) G satisfies the weak minimal condition on subgroups of infinite central dimension;

ii) G satisfies the weak maximal condition on subgroups of infinite central dimension;

iii) G satisfies the minimal condition on subgroups of infinite central dimension;

iv) G is Chernikov; and

v) G satisfies the minimal condition on all subgroups.

For non-periodic groups, the situation is more complicated. The study of locally nilpotent linear groups satisfying $Wmin - icd$ and $Wmax - icd$ has been initiated in the papers [20], [21]. The first result shows that nilpotent groups with these conditions are minimax.

Theorem 9. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$ having infinite central dimension. Suppose that H is a normal subgroup of G such that G/H is nilpotent. If G satisfies either $Wmin - icd$ or $Wmax - icd$, then G/H is minimax. In particular, if G is nilpotent, then G is minimax.*

Further results deal with to the case of prime characteristic.

Theorem 10. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let F be a field of prime characteristic, A a vector space over F and G a locally* *nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. if* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/Tor(G)$ *is minimax. In particular, if* $Tor(G)$ *has infinite central dimension, then* G *is minimax.*

Here $Tor(G)$ is the maximal normal periodic subgroup of G. If G is locally nilpotent group, then $Tor(G)$ consists of all elements of finite order, so that $G/Tor(G)$ is torsion-free.

Let $\mathcal{F}$ be the class of finite groups. If G is a group, then the intersection $G_{\mathcal{F}}$ of all subgroups of G, having finite index, is called the *finite reidual* of G.

Theorem 11. (L.A.Kurdachenko, J.M.Munoz-Escolano and J.Otal [20]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/G_{\mathcal{F}}$ *is minimax and nilpotent.*

Let $\mathcal{N}$ be the class of nilpotent groups. The intersection $G_{\mathcal{N}}$ of all normal subgroups H such that G/H is nilpotent, is called the *nilpotent residual* of G.

Theorem 12. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/G_{\mathcal{N}}$ *is minimax.*

For the case of non-finitary linear groups, the following results were obtained.

Theorem 13. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal, N.N. Semko [21]) *Let* F *be a field,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *is not finitary and satisfies* $Wmin - icd$*, then* G *is minimax.*

For the case of hypercentral groups and prime characteristic the study was completed. In fact, the following holds

Theorem 14. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal, N.N. Semko [21]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a hypercentral subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *nsatisfies* $Wmin - icd$*, then* G *is minimax.*

We observe that for the condition $Wmax - icd$ a similar result is not true. In the paper [21], a hypercentral linear group over the field of prime characteristic sarisfying $Wmax - icd$ which is not minimax was constructed.

The paper [22] began the study of soluble linear groups satisfying $Wmin - icd$. The following main result of this paper shows that their structure is rather similar to the structure of finite dimenional soluble groups.

Let $G \leq GL(F, A)$. We recall that an element $x \in G$ is called *unipotent* if there is a positive integer n such that $A(x - 1)^n = 0$. A subgroup H of G is called *unipotent* if every element of H is unipotent. A subgroup H of G is called *boundedly unipotent* if there is a positive integer n such that $A(x-1)^n = 0$ for each element $x \in H$.

Theorem 15. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [22]) *Let F be a field, A a vector space over F and G a soluble subgroup of $GL(F, A)$. Suppose that G has infinite central dimension and satisfies $Wmin - icd$. If G is not minimax, then G satisfies the following conditions:*

i) G has a normal boundedly unipotent subgroup L such that G/L is minimax;

ii) L has finite central dimension;

iii) if $charF = 0$, then L is nilpotent torsion-free subgroup;

iv) if $charF = p$ for some prime p, then L is a nilpotent bounded p-subgroup;

v) G is a finitary linear group.

If G is a subgroup of $GL(F, A)$, then G acts trivially on the factor-space $A/A(\omega FG)$. Hence G properly acts on the subspace $A(\omega FG)$. As in paper [23], we define the *augumentation dimension* of G to be the F-dimension of $A(\omega FG)$ and denote it by $augdim_F(G)$. This concept is opposite in some sense to the concept of central dimension. As for groups having finite central dimension, a group G of finite augmentation dimension contains a normal abelian subgroup C such that G/C is an ordinary finite dimensional group. Moreover, if $charF = 0$, then C is torsion-free, if $charF = p > 0$, then C is an elementary abelian p-subgroup. In the paper [23] linear groups in which the set of all subgroups having infinite augmentation dimension satisfies the minimal condition have been considered. In the paper [24] linear groups in which the set of all subgroups having infinite augmentation dimension satisfies some rank restrictions have been considered.

We can define finitary linear groups as the groups whose cyclic (and therefore finitely generated) subgroups have finite augmentation dimension. Therefore the following groups are the antipodes to finitary linear groups.

We say that a group $G \leq GL(F, A)$ is called *antifinitary linear group* if each proper infinitely generated subgroup of G has finite augmentation dimension (a subgroup H of an arbitrary group G is called *infinitely generated* if H has no a finite set of generators). These groups have been studied in the paper [25]. This study splits into two cases depending on whether or not the group is finitely generated.

Let $G \leq GL(F, A)$. Then the set

$$FD(G) = \{x \in G \quad | \quad < x > \quad \text{has finite augmentation dimension}\}$$

is a normal subgroup of G.

Let D be a divisible abelian group and G a subgroup of $Aut(D)$. Then D is said to be *G-divisibility irreducible* if D has no proper divisible G-invariant subgroups.

Theorem 16. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [25]) *Let F be a field, A a vector space over F and G a infinitely generated locally generalized radical subgroup of $GL(F, A)$. Suppose that G is not finitary and has infinite augmentation dimension. If G is not minimax, then G satisfies the following conditions:*

1) If the factor-group $G/FD(G)$ is infinitely generated, then G is a Prüfer p-group for some prime p.

2) If $G/FD(G)$ is finitely generated, then G satisfies the following conditions:

2a) $G = K < g >$ where K is a divisible abelian Chernikov subgroup and g is a p-element, where p is a prime such that $p = |G/FD(G)|$;

2b) K is a normal subgroup of G;

2c) K is G-divisibly irreducible;

2d) K is a q-subgroup for some prime q;

2f) if $q = p$, then K has finite special rank equal to $p^{m-1}(p-1)$ where $p^m = |< g > /C < g > (K)|$;

2g) if $q \neq p$, then K has finite special rank $o(q, p^m)$ where as above $p^m = |< g > /C < g > (K)|$ and $o(q, p^m)$ is the order of q modulo p^m.

Theorem 17. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [25]) *Let F be a field, A a vector space over F and G a finitely generated radical subgroup of $GL(F, A)$. Suppose that G is not finitary and has infinite augmentation dimension. Then the following conditions holds:*

1) $augdim_F FD(G)$ is finite;

2) G has a normal subgroup U such that G/U is polycyclic;

3) there is a positive integer m such that $A(x-1)m =< 0 >$ for each $x \in U$; in particular, U is nilpotent;

4) U is torsion-free if $char F = 0$ and is a bounded p-subgroup if $char F = p > 0$;

5) if

$$< 0 >= Z_0 \leq Z_1 \leq \cdots \leq Z_m = U \qquad (1)$$

is the upper central series of U, then $Z_1/Z_0, \dots, Z_m/Z_{m-1}$ are finitely generated $\mathbb{Z} < g >$-modules for each element $g \in G \backslash FD(G)$. In particular, U satisfies the maximal condition on $< g >$-invariant subgroups for each element $g \in G \backslash FD(G)$.

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